

Analytical formulas for calculating the extremal ranks of the matrix-valued function $A + BXC$ when the rank of X is fixed

Yongge Tian

CEMA, Central University of Finance and Economics, Beijing 100081, China

Abstract. One of the simplest matrix-valued function with a single variable matrix X is given by $A + BXC$. In this note, analytical formulas are established for calculating the maximal and minimal ranks of $A + BXC$ when the rank of the variable matrix X is fixed by using a simultaneous decomposition of A , B and C and some preliminary results. Some applications of the formulas in completing partially-specified block matrix with the maximal and minimal ranks are also given.

Keywords: Matrix-valued function; rank; objective function; feasible matrix set; optimization; simultaneous matrix decomposition

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1 Introduction

Fixed-rank or low-rank matrix approximation problems are to approximate optimally, with respect to some criteria, a matrix by one of the same dimension but fixed or smaller rank from a given feasible matrix set. Assume that A is a matrix to be approximated. Then a conventional statement of general matrix optimization problems of A from this point of view can be written as

$$\text{minimize } \rho(A - X) \quad \text{subject to } X \in \mathcal{S}, \quad (1.1)$$

where $\rho(\cdot)$ is a certain objective function of decision matrix, which is usually taken as determinant, trace, norms, rank, inertia of a matrix, and \mathcal{S} is a certain feasible matrix set. A best-known case of (1.1) is to minimize the norm $\|A - X\|_F^2$ subject to $X \in \mathcal{S}$. The fixed-rank or low-rank matrix set mentioned above can be written as

$$\mathcal{S} = \{X \mid \text{rank}(X) = t\} \quad \text{or} \quad \mathcal{S} = \{X \mid \text{rank}(X) \leq t\}. \quad (1.2)$$

The use of low-rank matrix to approximate a given matrix dates back to [2, 6], which now becomes a very active research subject in both optimization theory and applied disciplines.

Although these problems are stated quite clearly in form, it is hard in general to give satisfactory answers in closed-form to these matrix approximation problems. In other words, only numerical solutions to these approximation problems can be obtained. In this note, we assume that the objective function $\rho(\cdot)$ in (1.1) is taken as the rank of matrix. Then this kind of optimization problems can generally be written as

$$\text{maximize rank}(A - X) \quad \text{subject to } X \in \mathcal{S}, \quad (1.3)$$

$$\text{minimize rank}(A - X) \quad \text{subject to } X \in \mathcal{S}, \quad (1.4)$$

respectively. The rank of matrix, as an objective function, is often used when finding feasible matrix X such that resulting $A - X$ attains its maximal possible rank (is nonsingular when square), or such that $A - X$ attains the minimal rank as possible (called low-rank matrix completion). This kind of problems are usually called the matrix rank-maximization and rank-minimization problems, or matrix rank completion problems in the literature. Generally speaking, matrix rank-optimization problems are a class of discontinuous optimization problems, in which the decision variables are matrices running over certain matrix sets, while the ranks of the variable matrices are taken as integer-valued objective functions. In this case, analytical formulas for calculating the integer extremum ranks of $A - X$ can hardly be derived by numerical approximation methods. This fact means that solving methods of matrix rank optimization problems are not consistent with any of the ordinary continuous and discrete problems in optimization theory, so that we cannot apply various common methods of solving continuous optimization problems, such as the well-known differential and Lagrangian methods, to approach these constrained optimization problems. Instead, we can only find the exact maximal and minimal ranks through pure algebraic operations of matrices. It has been known that matrix rank-optimization problems are NP-hard in general due to the discontinuity and combinational nature of rank of a matrix and the algebraic structure of \mathcal{S} . Many new researches were conducted on this

E-mail Address: yongge.tian@gmail.com

kind of matrix rank-optimization problems from theory and applied points of view in the past decades; see, e.g., [5]. Because the rank of a matrix can only take finite integers between 0 and the dimensions of the matrix, it is really expected to establish certain analytical formulas for calculating the maximal and minimal ranks for curiosity.

In what follows, we assume that $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times p}$ and $C \in \mathbb{C}^{q \times n}$ are given matrices, and the feasible matrix set \mathcal{S} in (1.1) is taken as

$$\mathcal{S} = \{-BXC \mid X \in \mathbb{C}^{p \times q} \text{ and } \text{rank}(X) = t\}. \quad (1.5)$$

Then, the difference in (1.1) can equivalently be written as the following linear matrix-valued function

$$\phi(X) = A + BXC, \quad (1.6)$$

which is a map $\phi : \mathbb{C}^{p \times q} \rightarrow \mathbb{C}^{m \times n}$. Under such a formulation, this note aims at solving the following constrained matrix optimization problems:

Problem 1.1 For the function in (1.6) and two integers s and t with $0 \leq s \leq t \leq \min\{p, q\}$, establish explicit formulas for calculating the following extremal ranks

$$\text{maximize } \text{rank}(A + BXC) \quad \text{s.t. } X \in \mathbb{C}^{p \times q} \text{ and } \text{rank}(X) = t, \quad (1.7)$$

$$\text{minimize } \text{rank}(A + BXC) \quad \text{s.t. } X \in \mathbb{C}^{p \times q} \text{ and } \text{rank}(X) = t, \quad (1.8)$$

$$\text{maximize } \text{rank}(A + BXC) \quad \text{s.t. } X \in \mathbb{C}^{p \times q} \text{ and } s \leq \text{rank}(X) \leq t, \quad (1.9)$$

$$\text{minimize } \text{rank}(A + BXC) \quad \text{s.t. } X \in \mathbb{C}^{p \times q} \text{ and } s \leq \text{rank}(X) \leq t. \quad (1.10)$$

The matrices X satisfying the constraints in (1.7)–(1.10) are called the feasible solutions (i.e., candidates for solutions) to the problems, respectively. They form certain sets of $\mathbb{C}^{p \times q}$ and it is over these sets that the objective function is to be maximized or minimized. However, these matrix sets are not necessarily convex. Motivations for finding the extremal ranks of (1.6) arise from both theoretical and applied points of view. It is really lucky that we can establish analytical formulas for calculating the extremal ranks of matrix-valued functions for some special matrix sets \mathcal{S} by using various expansion formulas for ranks of matrices and some tricky matrix operations. For instance, two well-known seminal formulas in closed-form for calculating the global maximal and minimal ranks of (1.6) are given by

$$\max_{X \in \mathbb{C}^{p \times q}} \text{rank}(A + BXC) = \min \left\{ \text{rank}[A, B], \text{rank} \begin{bmatrix} A \\ C \end{bmatrix} \right\}, \quad (1.11)$$

$$\min_{X \in \mathbb{C}^{p \times q}} \text{rank}(A + BXC) = \text{rank}[A, B] + \text{rank} \begin{bmatrix} A \\ C \end{bmatrix} - \text{rank} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \quad (1.12)$$

Because the right-hand sides of (1.11) and (1.12) are calculated only by three block matrices composed by the three given matrices, a beginner who knows the concept of matrix rank in linear algebra can understand the usefulness of (1.11) and (1.12). People can apply (1.11) and (1.12) to characterize many fundamental behaviors of $A + BXC$, for instance, necessary and sufficient conditions can directly be established for $A + BXC$ to be nonsingular; for $A + BXC$ to be zero; for the rank of $A + BXC$ to be invariant under different choice of X ; for the row and column spaces of $A + BXC$ to be invariant under different choice of X , respectively, etc. However, these two elementary formulas cannot be proved within the scope of elementary linear algebra. Some people made essential contributions for the establishments of (1.11) and (1.12) through pure algebraic operations of the given matrices and generalized inverses, as well as simultaneous matrix decompositions of the given matrices; see, e.g., [1, 4, 8, 9]. Analytical expressions for the general expressions of the variable matrices X satisfying (1.11) and (1.12) were also obtained through generalized inverses and simultaneous matrix decompositions of the given matrices in [4, 9]. Eqs. (1.11) and (1.12) are not just two isolated formulas for the maximal and minimal ranks of matrix-valued functions. Motivated by some recent work on low-rank matrix approximations, the present author revisits (1.6) by adding certain rank restrictions on the variable matrix X , and establishes some new and elementary formulas for calculating the maximal and minimal ranks in (1.7)–(1.10), which, we believe, can be taken as some standard examples for verifying accuracy of various algorithms in solving matrix rank-approximation problems.

Throughout this note, $\mathbb{C}^{m \times n}$ stands for the set of all $m \times n$ complex matrices; $\mathbb{C}_t^{m \times n}$ stands for the set of all $m \times n$ complex matrices with $\text{rank}(X) = t$; A^* , $r(A)$ and $\mathcal{R}(A)$ stand for the conjugate transpose, rank and range (column space) of a matrix $A \in \mathbb{C}^{m \times n}$, respectively; I_m denotes the identity matrix of order m ; $[A, B]$ denotes a row block matrix consisting of A and B .

In dealing with problems in the formats of (1.7)–(1.10), people usually construct certain canonical forms of the matrix-valued functions through some simultaneous decompositions of A , B and C , because the ranks of matrices are invariant under nonsingular matrix transformations. In order to establish a canonical form of (1.6), we need the following several known or simple results on simultaneous decompositions of matrices and rank formulas for block matrices.

Lemma 1.2 ([10, 11]) *Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times p}$ and $C \in \mathbb{C}^{q \times n}$. Then there exist two nonsingular matrices $P \in \mathbb{C}^{m \times m}$, $Q \in \mathbb{C}^{n \times n}$ and two unitary matrices $U \in \mathbb{C}^{p \times p}$, $V \in \mathbb{C}^{q \times q}$ such that*

$$A = P\Sigma_A Q, \quad B = P\Sigma_B U, \quad C = V\Sigma_C Q, \quad (1.13)$$

where

$$\Sigma_A = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & S_A & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} j \\ k \\ l \\ u \\ s_2 \\ t_2 \end{matrix}, \quad (1.14)$$

$j \quad k \quad l \quad u \quad s_1 \quad t_1$

$$\Sigma_B = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & S_B & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} j \\ k+l \\ u \\ s_2 \\ t_2 \end{matrix}, \quad (1.15)$$

$j \quad p-j-r-s_2 \quad u \quad s_2$

$$\Sigma_C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & S_C & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} q-l-u-s_1 \\ l \\ u \\ s_1 \\ t_1 \end{matrix}, \quad (1.16)$$

$j+k \quad l \quad u \quad s_1 \quad t_1$

S_A , S_B and S_C are diagonal matrices with positive diagonal entries, and

$$\begin{aligned} j &= r \begin{bmatrix} A \\ C \end{bmatrix} + r(B) - r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \\ k &= r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(B) - r(C), \\ l &= r[A, B] + r(C) - r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \\ u &= r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + r(A) - r \begin{bmatrix} A \\ C \end{bmatrix} - r[A, B], \\ s_1 &= r \begin{bmatrix} A \\ C \end{bmatrix} - r(A), \\ s_2 &= r[A, B] - r(A), \\ t_1 &= n - r \begin{bmatrix} A \\ C \end{bmatrix}, \\ t_2 &= m - r[A, B]. \end{aligned}$$

Lemma 1.3 *Let $X \in \mathbb{C}^{m \times n}$, $Y \in \mathbb{C}^{m \times p}$ and $Z \in \mathbb{C}^{q \times n}$ be three variable matrices, and let*

$$\phi(X, Y, Z) = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix}. \quad (1.17)$$

Then,

$$\max_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}^{m \times p}, Z \in \mathbb{C}^{q \times n}} r[\phi(X, Y, Z)] = \min\{m+q, \quad n+p, \quad m+n\}, \quad (1.18)$$

$$\min_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}^{m \times p}, Z \in \mathbb{C}^{q \times n}} r[\phi(X, Y, Z)] = 0. \quad (1.19)$$

Further, for any integer t with $0 \leq t \leq \min\{m+q, n+p, m+n\}$, there exist $X \in \mathbb{C}^{m \times n}$, $Y \in \mathbb{C}^{m \times p}$ and $Z \in \mathbb{C}^{q \times n}$ such that

$$r \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix} = t. \quad (1.20)$$

Proof. It is obvious that the right-hand side of (1.18) is an upper bound of $r[\phi(X, Y, Z)]$.

(I) Under $m+q \leq \min\{n+p, m+n\}$ and $m \leq p$, setting

$$X = 0, \quad Y = [I_m, 0], \quad Z = [I_q, 0]$$

leads to $r[\phi(X, Y, Z)] = r(Y) + r(Z) = m+q$; under $m+q \leq \min\{n+p, m+n\}$ and $m > p$, setting

$$[X, Y] = [0, I_m], \quad Z = [I_q, 0]$$

leads to $r[\phi(X, Y, Z)] = m+q$;

(II) under $n+p \leq \min\{m+q, m+n\}$ and $n \leq q$, setting

$$X = 0, \quad Y = [I_n, 0]^T, \quad Z = [I_p, 0]^T$$

leads to $r[\phi(X, Y, Z)] = r(Y) + r(Z) = n+p$; under $n+p \leq \min\{m+q, m+n\}$ and $n > q$, setting

$$\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ I_n \end{bmatrix}, \quad Y = \begin{bmatrix} I_p \\ 0 \end{bmatrix}$$

leads to $r[\phi(X, Y, Z)] = r(Y) + r(Z) = n+p$;

(III) under $m+n \leq \min\{m+q, n+p\}$, setting

$$X = 0, \quad Y = [I_m, 0], \quad Z = [I_n, 0]^T$$

leads to $r[\phi(X, Y, Z)] = r(Y) + r(Z) = m+n$; establishing (1.18).

Setting $X = 0$ and $Y = 0$ leads to (1.19).

(a) for any integer $0 \leq t \leq \min\{m+q, n+p, m+n\}$ with $m+q \leq \min\{n+p, m+n\}$ and $m \leq p$, setting

$$X = 0, \quad Y = [Y_1, 0], \quad Z = [Z_1, 0], \quad r(Y_1) + r(Z_1) = t$$

leads to $r[\phi(X, Y, Z)] = r(Y_1) + r(Z_1) = t$; with $m+q \leq \min\{n+p, m+n\}$ and $m > p$, setting

$$[X, Y] = [0, Y_1], \quad Z = [Z_1, 0], \quad r(Y_1) + r(Z_1) = t$$

leads to $r[\phi(X, Y, Z)] = r(Y_1) + r(Z_1) = t$;

(b) for any integer $0 \leq t \leq \min\{m+q, n+p, m+n\}$ with $n+p \leq \min\{m+q, m+n\}$ and $n \leq q$, setting

$$X = 0, \quad Y = [Y_1, 0]^T, \quad Z = [Z_1, 0]^T, \quad r(Y_1) + r(Z_1) = t$$

leads to $r[\phi(X, Y, Z)] = r(Y_1) + r(Z_1) = t$; with $n+p \leq \min\{m+q, m+n\}$ and $n > q$, setting

$$\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ Z_1 \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 \\ 0 \end{bmatrix}$$

leads to $r[\phi(X, Y, Z)] = r(Y_1) + r(Z_1) = t$;

(c) for any integer $0 \leq t \leq \min\{m+q, n+p, m+n\}$ with $m+n \leq \min\{m+q, n+p\}$, setting

$$X = 0, \quad Y = [Y_1, 0], \quad Z = [Z_1, 0]^T, \quad r(Y_1) + r(Z_1) = t$$

leads to $r[\phi(X, Y, Z)] = r(Y_1) + r(Z_1) = t$, establishing (1.20). \square

Lemma 1.4 Let $A \in \mathbb{C}^{m \times n}$ be given, $Y \in \mathbb{C}^{m \times p}$, $Z \in \mathbb{C}^{q \times n}$ and $U \in \mathbb{C}^{q \times p}$ be three variable matrices, and define

$$\phi(Y, Z, U) = \begin{bmatrix} A & Y \\ Z & U \end{bmatrix}. \quad (1.21)$$

Then,

$$\max_{Y \in \mathbb{C}^{m \times p}, Z \in \mathbb{C}^{q \times n}, U \in \mathbb{C}^{q \times p}} r[\phi(X, Y, U)] = \min\{m + p, n + q, p + q - r(A)\}, \quad (1.22)$$

$$\min_{Y \in \mathbb{C}^{m \times p}, Z \in \mathbb{C}^{q \times n}, U \in \mathbb{C}^{q \times p}} r[\phi(X, Y, U)] = r(A). \quad (1.23)$$

In particular, for any integer t with $r(A) \leq t \leq \min\{m + p, n + q, r(A) + p + q\}$, there exist $Y \in \mathbb{C}^{m \times p}$, $Z \in \mathbb{C}^{q \times n}$ and $U \in \mathbb{C}^{q \times p}$ such that

$$r[\phi(X, Y, U)] = t. \quad (1.24)$$

Proof. Without lost generality, we assume that A is given by

$$A = \text{diag}(I_d, 0). \quad (1.25)$$

Correspondingly,

$$\phi(Y, Z, U) = \begin{bmatrix} I_d & 0 & \hat{Y}_1 \\ 0 & 0 & \hat{Y}_2 \\ \hat{Z}_1 & \hat{Z}_2 & U \end{bmatrix}, \quad (1.26)$$

and

$$r[\phi(Y, Z, U)] = d + r \begin{bmatrix} 0 & \hat{Y}_2 \\ \hat{Z}_2 & U - \hat{Z}_1 \hat{Y}_1 \end{bmatrix}. \quad (1.27)$$

Applying (1.18) and (1.19) to the block matrix in (1.27) leads to

$$\max_{Y \in \mathbb{C}^{m \times p}, Z \in \mathbb{C}^{q \times n}, U \in \mathbb{C}^{q \times p}} r \begin{bmatrix} 0 & \hat{Y}_2 \\ \hat{Z}_2 & U - \hat{Z}_1 \hat{Y}_1 \end{bmatrix} = \min\{m + p - r(A), n + q - r(A), p + q - 2r(A)\}, \quad (1.28)$$

$$\min_{Y \in \mathbb{C}^{m \times p}, Z \in \mathbb{C}^{q \times n}, U \in \mathbb{C}^{q \times p}} r \begin{bmatrix} 0 & \hat{Y}_2 \\ \hat{Z}_2 & U - \hat{Z}_1 \hat{Y}_1 \end{bmatrix} = 0. \quad (1.29)$$

Substituting (1.28) and (1.29) into (1.27) yields (1.22) and (1.23). Applying (1.20) to (1.28) and (1.29) leads to (1.24). \square

2 Rank optimization of $A + X$

One of the special cases in (1.6) is the ordinary sum $A + X$. In this section, we derive explicit formulas for calculating the extremal ranks of $A + X$ subject to X with a fixed rank. The formulas obtained will be used in Sections 3.

Theorem 2.1 *Let $A \in \mathbb{C}^{m \times n}$ be given, $X \in \mathbb{C}^{m \times n}$ be a variable matrix, and assume that s and t are two integers satisfying*

$$0 \leq s \leq t \leq \min\{m, n\}. \quad (2.1)$$

Then,

(a) *The following equalities hold*

$$\max_{X \in \mathbb{C}_t^{m \times n}} r(A + X) = \min\{m, n, r(A) + t\}, \quad (2.2)$$

$$\min_{X \in \mathbb{C}_t^{m \times n}} r(A + X) = |r(A) - t|. \quad (2.3)$$

(b) *The following equalities hold*

$$\max_{X \in \mathbb{C}^{m \times n}, s \leq r(X) \leq t} r(A + X) = \min\{m, n, r(A) + t\}, \quad (2.4)$$

$$\min_{X \in \mathbb{C}^{m \times n}, s \leq r(X) \leq t} r(A + X) = \max\{0, s - r(A), r(A) - t\}. \quad (2.5)$$

(c) The following equalities hold

$$\max_{X \in \mathbb{C}^{m \times n}, 0 \leq r(X) \leq t} r(A + X) = \min\{m, n, r(A) + t\}, \quad (2.6)$$

$$\min_{X \in \mathbb{C}^{m \times n}, 0 \leq r(X) \leq t} r(A + X) = \max\{0, r(A) - t\}. \quad (2.7)$$

(d) The following equalities hold

$$\max_{X \in \mathbb{C}^{m \times n}, s \leq r(X) \leq \min\{m, n\}} r(A + X) = \min\{m, n\}, \quad (2.8)$$

$$\min_{X \in \mathbb{C}^{m \times n}, s \leq r(X) \leq \min\{m, n\}} r(A + X) = \max\{0, s - r(A)\}. \quad (2.9)$$

The matrices X satisfying these equalities can be formulated from the canonical form of A .

Proof. It is obvious that the right-hand sides of (2.2) and (2.3) are upper and lower bounds. Without loss of generality, we assume that A is of the form

$$A = \text{diag}(I_d, 0). \quad (2.10)$$

Let $X = \begin{bmatrix} 0 & 0 \\ 0 & I_t \end{bmatrix}$. If $m \leq \min\{n, r(A) + t\}$, then $r(A + X) = m$; if $n \leq \min\{m, r(A) + t\}$, then $r(A + X) = n$; if $r(A) + t \leq \min\{m, n\}$, then $r(A + X) = r(A) + t$, so that (2.2) holds.

If $r(A) \leq t$, then setting $X = \begin{bmatrix} -I_d & 0 & 0 \\ 0 & I_{t-d} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ gives $r(A + X) = t - d$; if $r(A) > t$, then setting $X = \begin{bmatrix} -I_t & 0 \\ 0 & 0 \end{bmatrix}$ gives $r(A + X) = d - t$, so that (2.3) holds.

Note that

$$\{X \in \mathbb{C}^{m \times n} \mid s \leq r(X) \leq t\} = \mathbb{C}_s^{m \times n} \cup \mathbb{C}_{s+1}^{m \times n} \cup \dots \cup \mathbb{C}_t^{m \times n}.$$

So that

$$\max_{X \in \mathbb{C}^{m \times n}, s \leq r(X) \leq t} r(A + X) = \max \left\{ \max_{X \in \mathbb{C}_s^{m \times n}} r(A + X), \max_{X \in \mathbb{C}_{s+1}^{m \times n}} r(A + X), \dots, \max_{X \in \mathbb{C}_t^{m \times n}} r(A + X) \right\}, \quad (2.11)$$

$$\min_{X \in \mathbb{C}^{m \times n}, s \leq r(X) \leq t} r(A + X) = \min \left\{ \min_{X \in \mathbb{C}_s^{m \times n}} r(A + X), \min_{X \in \mathbb{C}_{s+1}^{m \times n}} r(A + X), \dots, \min_{X \in \mathbb{C}_t^{m \times n}} r(A + X) \right\}. \quad (2.12)$$

Substituting (2.2) and (2.3) for $r(X) = s, s + 1, \dots, t$ into (2.11) and (2.12) and making the max-min comparison, we obtain

$$\begin{aligned} & \max_{X \in \mathbb{C}^{m \times n}, s \leq r(X) \leq t} r(A + X) \\ &= \max \{ \min\{m, n, r(A) + s\}, \min\{m, n, r(A) + s + 1\}, \dots, \min\{m, n, r(A) + t\} \} \\ &= \min\{m, n, r(A) + t\}, \end{aligned} \quad (2.13)$$

$$\begin{aligned} & \min_{X \in \mathbb{C}^{m \times n}, s \leq r(X) \leq t} r(A + X) \\ &= \min \{ |r(A) - s|, |r(A) - s - 1|, \dots, |r(A) - t| \} = \max\{0, s - r(A), r(A) - t\}, \end{aligned} \quad (2.14)$$

establishing (2.4) and (2.5), as well as (2.6)–(2.9). \square

The results in the section show that the matrix rank optimization formulated in (1.7)–(1.10) are combinatorial in nature.

3 Rank optimization of $A + BXC$

A matrix-valued function for complex matrices is a map between matrix spaces $\mathbb{C}^{m \times n}$ and $\mathbb{C}^{p \times q}$, which can generally be written as

$$Y = f(X) \text{ for } Y \in \mathbb{C}^{m \times n} \text{ and } X \in \mathbb{C}^{p \times q},$$

or briefly, $f : \mathbb{C}^{m \times n} \rightarrow \mathbb{C}^{p \times q}$. Eq. (1.6) is in fact the simplest case of all matrix-valued functions, which is extensively studied from theoretical and applied points of view.

According to [10], substituting (1.13) into (1.6) yields

$$\phi(X) = P\Sigma_A Q + P\Sigma_B U X V \Sigma_C Q = P(\Sigma_A + \Sigma_B U X V \Sigma_C)Q, \quad (3.1)$$

which we call a canonical form of (1.6). Many properties of the matrix-valued function $A + BXC$ can be derived from the canonical form. For instance, the rank of $A + BXC$ is

$$r(A + BXC) = r(\Sigma_A + \Sigma_B Y \Sigma_C), \quad r(X) = r(Y), \quad (3.2)$$

where $Y = UXV$. Partition it as

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} \begin{matrix} j \\ p-j-u-s_2 \\ u \\ s_2 \end{matrix} \\ \begin{matrix} q-l-u-s_1 & l & u & s_1 \end{matrix}. \quad (3.3)$$

Then we have

$$\Sigma_A + \Sigma_B Y \Sigma_C = \begin{bmatrix} I_j & 0 & Y_{12} & Y_{13} S_C & Y_{14} & 0 \\ 0 & I_k & 0 & 0 & 0 & 0 \\ 0 & 0 & I_l & 0 & 0 & 0 \\ 0 & 0 & S_B Y_{32} & S_A + S_B Y_{33} S_C & S_B Y_{34} & 0 \\ 0 & 0 & Y_{42} & Y_{43} S_C & Y_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (3.4)$$

and

$$\begin{aligned} r(\Sigma_A + \Sigma_B Y \Sigma_C) &= j + k + l + r \begin{bmatrix} S_A + S_B Y_{33} S_C & S_B Y_{34} \\ Y_{43} S_C & Y_{44} \end{bmatrix} \\ &= r[A, B] + r \begin{bmatrix} A \\ C \end{bmatrix} - r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + r \begin{bmatrix} S_B^{-1} S_A S_C^{-1} + Y_{33} & Y_{34} \\ Y_{43} & Y_{44} \end{bmatrix}. \end{aligned} \quad (3.5)$$

So that

$$\max_{Y \in \mathbb{C}_t^{p \times q}} r(\Sigma_A + \Sigma_B Y \Sigma_C) = r[A, B] + r \begin{bmatrix} A \\ C \end{bmatrix} - r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + \max_{Y \in \mathbb{C}_t^{p \times q}} r(S + \hat{Y}), \quad (3.6)$$

$$\min_{Y \in \mathbb{C}_t^{p \times q}} r(\Sigma_A + \Sigma_B Y \Sigma_C) = r[A, B] + r \begin{bmatrix} A \\ C \end{bmatrix} - r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + \min_{Y \in \mathbb{C}_t^{p \times q}} r(S + \hat{Y}), \quad (3.7)$$

where

$$S = \begin{bmatrix} S_B^{-1} S_A S_C^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{Y} = \begin{bmatrix} Y_{33} & Y_{34} \\ Y_{43} & Y_{44} \end{bmatrix}.$$

Applying Theorem 2.1(b) to (3.6) and (3.7) yields the main result in the note.

Theorem 3.1 *Let $\phi(X)$ be as given in (1.6), and assume that t is an integer satisfying $0 \leq t \leq \min\{p, q\}$. Also define*

$$G = [A, B], \quad H = \begin{bmatrix} A \\ C \end{bmatrix}, \quad M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \quad (3.8)$$

Then,

$$\max_{X \in \mathbb{C}_t^{p \times q}} r(A + BXC) = \min \left\{ r[A, B] + r \begin{bmatrix} A \\ C \end{bmatrix}, \quad r(A) + t \right\}, \quad (3.9)$$

$$\min_{X \in \mathbb{C}_t^{p \times q}} r(A + BXC) = \max \left\{ r[A, B] + r \begin{bmatrix} A \\ C \end{bmatrix} - r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \quad r[A, B] + r \begin{bmatrix} A \\ C \end{bmatrix} - r(A) + t - p - q, \quad r(A) - t \right\}. \quad (3.10)$$

In consequences,

(a) Under $m = n$, there exists an $X \in \mathbb{C}_t^{p \times q}$ such that $A + BXC$ is nonsingular if and only if

$$r(G) = r(H) = m \text{ and } r(A) \geq m - t. \quad (3.11)$$

(b) There exists an $X \in \mathbb{C}_t^{p \times q}$ such that $A + BXC = 0$ if and only if

$$\mathcal{R}(A) \subseteq \mathcal{R}(B), \quad \mathcal{R}(A^*) \subseteq \mathcal{R}(C^*), \quad r(G) + r(H) \leq r(A) - t + p + q, \quad r(A) \leq t. \quad (3.12)$$

(c) Under $t \neq 0$, the rank of $A + BXC$ is invariant for all $X \in \mathbb{C}_t^{p \times q}$ if and only if one of the following conditions holds:

- (i) $r(M) = r(G)$,
- (ii) $r(M) = r(H)$,
- (iii) $r(M) = r(G) + f(H) - r(A) - t$,
- (iv) $r(G) = r(A) + p + q - t$,
- (v) $r(H) = r(A) + p + q - t$,
- (vi) $r(G) + r(H) = 2r(A) + p + q$, namely, $r(B) = p$, $r(C) = q$, $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ and $\mathcal{R}(A^*) \cap \mathcal{R}(C^*) = \{0\}$.

(d) Under $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(C^*)$,

$$\max_{X \in \mathbb{C}_t^{p \times q}} r(A + BXC) = \min \{r(B), \quad r(C), \quad r(A) + t\}, \quad (3.13)$$

$$\min_{X \in \mathbb{C}_t^{p \times q}} r(A + BXC) = \max \{0, \quad r(B) + r(C) - r(A) + t - p - q, \quad r(A) - t\}. \quad (3.14)$$

Proof. Let $z = p + q - 2r(M) + r(G) + r(H)$. Then we find by (1.24), (2.4) and (2.5) that

$$\begin{aligned} \max_{Y \in \mathbb{C}_t^{p \times q}} r(S + \hat{Y}) &= \max_{t - z \leq r(\hat{Y}) \leq t} r(S + \hat{Y}) = \min \{u + s_1, \quad u + s_2, \quad r(S) + t\} \\ &= \min \{r(M) - r(G), \quad r(M) - r(H), \quad r(M) + r(A) - r(G) - r(H) + t\}, \end{aligned} \quad (3.15)$$

$$\begin{aligned} \min_{Y \in \mathbb{C}_t^{p \times q}} r(S + \hat{Y}) &= \min_{t - z \leq r(\hat{Y}) \leq t} r(S + \hat{Y}) = \max \{0, \quad t - z - r(S), \quad r(S) - t\} \\ &= \max \{0, \quad t - p - q + r(M) - r(A), \quad r(M) + r(A) - r(G) - r(H) - t\}. \end{aligned} \quad (3.16)$$

Substituting (3.15) and (3.16) into (3.6) and (3.7) yields (3.9) and (3.10). Setting (3.9) equal to m yields (3.11); setting (3.10) equal to 0 yields (3.12); setting (3.9) equal to (3.10) yields the results in (c). \square

Eqs. (3.9) and (3.10) show that the extremal ranks of (1.6) can be calculated exactly from the two formulas without knowing how to choose the feasible matrices X . So that they can be used independently in describing behaviors of $A + BXC$, as shown in Theorem 3.1(a)–(d).

Recall that any matrix $X \in \mathbb{C}_t^{p \times q}$ can be written as a product $X = YZ$, where $Y \in \mathbb{C}_t^{p \times t}$ and $Z \in \mathbb{C}_t^{t \times q}$ with $r(Y) = r(Z) = t$. So that (3.9) and (3.10) can be represented as follows.

Corollary 3.2 Let $\phi(X)$ be as given in (1.6), t be an integer satisfying $0 \leq t \leq \min\{p, q\}$, and G, H and M be the matrices in (3.8). Then,

$$\max_{Y \in \mathbb{C}_t^{p \times t}, Z \in \mathbb{C}_t^{t \times q}} r(A + BYZC) = \min \{r(G), \quad r(H), \quad r(A) + t\}, \quad (3.17)$$

$$\min_{Y \in \mathbb{C}_t^{p \times t}, Z \in \mathbb{C}_t^{t \times q}} r(A + BYZC) = \max \{r(G) + r(H) - r(M), \quad r(G) + r(H) - r(A) + t - p - q, \quad r(A) - t\}. \quad (3.18)$$

Corollary 3.3 Let $\phi(X)$ be as given in (1.6), G, H and M be the matrices in (3.8), and assume that s and t are two integers satisfying

$$0 \leq s \leq t \leq \min\{p, q\}. \quad (3.19)$$

Then,

$$\max_{X \in \mathbb{C}^{p \times q}, s \leq r(X) \leq t} r(A + BXC) = \min \{r(G), \quad r(H), \quad r(A) + t\}, \quad (3.20)$$

$$\min_{X \in \mathbb{C}^{p \times q}, s \leq r(X) \leq t} r(A + BXC) = \min \{u_s, \quad u_{s+1}, \dots, u_t\}, \quad (3.21)$$

where

$$u_l = \max \{r(G) + r(H) - r(M), \quad r(G) + r(H) - r(A) + l - p - q, \quad r(A) - l\}, \quad l = s, s+1, \dots, t.$$

Corollary 3.4 Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times p}$ and $C \in \mathbb{C}^{p \times n}$ be given. Then,

$$\max_{X \in \mathbb{C}_p^{p \times p}} r(A + BXC) = \min \left\{ r[A, B], \quad r \begin{bmatrix} A \\ C \end{bmatrix} \right\}, \quad (3.22)$$

$$\min_{X \in \mathbb{C}_p^{p \times p}} r(A + BXC) = \max \left\{ r[A, B] + r \begin{bmatrix} A \\ C \end{bmatrix} - r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \quad r[A, B] + r \begin{bmatrix} A \\ C \end{bmatrix} - r(A) - p \right\}. \quad (3.23)$$

Corollary 3.5 Let $0 \neq B \in \mathbb{C}^{m \times p}$ and $0 \neq C \in \mathbb{C}^{q \times n}$ be given, and assume that t is an integer satisfying $1 \leq t \leq \min\{p, q\}$. Then,

$$\max_{X \in \mathbb{C}^{p \times q}, r(X)=t} r(BXC) = \min \{r(B), \quad r(C), \quad t\}, \quad (3.24)$$

$$\min_{X \in \mathbb{C}^{p \times q}, r(X)=t} r(BXC) = \max \{0, \quad r(B) + r(C) + t - p - q\}. \quad (3.25)$$

In consequences,

(i) Under $m = n$, there exists an $X \in \mathbb{C}^{p \times q}$ with $r(X) = t$ such that BXC is nonsingular if and only if

$$r(B) = r(C) = m \quad \text{and} \quad t \geq m. \quad (3.26)$$

(ii) There exists an $X \in \mathbb{C}^{p \times q}$ with $r(X) = t$ such that $BXC = 0$ if and only if

$$r(B) + r(C) \leq p + q - t. \quad (3.27)$$

(iii) The rank of BXC is invariant for all $X \in \mathbb{C}^{p \times q}$ with $r(X) = t$ if and only if

$$r(B) = p + q - t, \quad \text{or} \quad r(C) = p + q - t, \quad \text{or} \quad r(B) = p \text{ and } r(C) = q. \quad (3.28)$$

4 Completing a partially-specified block matrix with extremal ranks

As an application of the results in the previous section, we consider the rank of the following partially specified block matrix

$$\phi(X) = \begin{bmatrix} A & B \\ C & X \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_q \end{bmatrix} X [0, I_p], \quad (4.1)$$

where $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times p}$ and $C \in \mathbb{C}^{q \times n}$ are given, and $X \in \mathbb{C}^{q \times p}$ is a variable matrix, which obviously is a special case of (1.6). Conversely, the rank of (1.6) can equivalently be written as

$$r(A + BXC) = r \begin{bmatrix} A & B & 0 \\ C & 0 & I_q \\ 0 & I_p & X \end{bmatrix} - p - q,$$

the block matrix in which is a special case of (4.1) as well.

Theorem 4.1 Let $\phi(X)$ be as given in (4.1), and assume that s and t are two integers satisfying

$$0 \leq s \leq t \leq \min\{p, q\}. \quad (4.2)$$

Also define

$$G = [A, B], \quad H = \begin{bmatrix} A \\ C \end{bmatrix}, \quad M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \quad (4.3)$$

Then,

(a) *The following equalities hold*

$$\max_{X \in \mathbb{C}_t^{q \times p}} r[\phi(X)] = \min \{r(G) + q, \quad r(H) + p, \quad r(M) + t\}, \quad (4.4)$$

$$\min_{X \in \mathbb{C}_t^{q \times p}} r[\phi(X)] = \max \{r(G) + r(H) - r(A), \quad r(G) + r(H) - r(M) + t, \quad r(M) - t\}. \quad (4.5)$$

In consequence,

- (i) *Under $m + q = n + p$, there exists an $X \in \mathbb{C}_t^{q \times p}$ such that $\phi(X)$ is nonsingular if and only if $r(G) = m$, $r(H) = n$ and $r(M) \geq m + q - t$.*
- (ii) *Under $m + q = n + p$, $\phi(X)$ is nonsingular for all $X \in \mathbb{C}_t^{q \times p}$ if and only if $r(G) + r(H) - r(A) = m + q$ or $r(G) + r(H) - r(M) = m + q - t$, or $r(M) = m + q$ and $t = 0$.*

(b) *Under $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(A^*)$, the following equalities hold*

$$\max_{X \in \mathbb{C}_t^{q \times p}} r[\phi(X)] = \min \{r(A) + q, \quad r(A) + p, \quad r(M) + t\}, \quad (4.6)$$

$$\min_{X \in \mathbb{C}_t^{q \times p}} r[\phi(X)] = \max \{r(A), \quad 2r(A) - r(M) + t, \quad r(M) - t\}. \quad (4.7)$$

(c) *Under $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(C^*)$, the following equalities hold*

$$\max_{X \in \mathbb{C}_t^{q \times p}} r[\phi(X)] = \min \{r(B) + q, \quad r(C) + p, \quad r(B) + r(C) + t\}, \quad (4.8)$$

$$\min_{X \in \mathbb{C}_t^{q \times p}} r[\phi(X)] = \max \{r(B) + r(C) - r(A), \quad t, \quad r(B) + r(C) - t\}. \quad (4.9)$$

(d) *The following equalities hold*

$$\max_{X \in \mathbb{C}^{q \times p}, s \leq r(X) \leq t} r[\phi(X)] = \min \{r(G) + q, \quad r(H) + p, \quad r(M) + t\}, \quad (4.10)$$

$$\min_{X \in \mathbb{C}^{q \times p}, s \leq r(X) \leq t} r[\phi(X)] = \min \{u_s, \quad u_{s+1}, \dots, u_t\}, \quad (4.11)$$

where

$$u_l = \max \{r(G) + r(H) - r(A), \quad r(G) + r(H) - r(M) + l, \quad r(A) - l\}, \quad l = s, s + 1, \dots, t.$$

Theorem 4.2 *Let*

$$\phi(X) = \begin{bmatrix} A - X & B - X \\ C - X & D - X \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} I_m \\ I_m \end{bmatrix} X \begin{bmatrix} I_n & I_n \end{bmatrix}, \quad (4.12)$$

where $A, B, C, D \in \mathbb{C}^{m \times n}$. Also define

$$G = [A - C, B - D], \quad H = \begin{bmatrix} A - B \\ C - D \end{bmatrix}, \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (4.13)$$

Then the following equalities hold

$$\max_{X \in \mathbb{C}_t^{m \times n}} r[\phi(X)] = \min \{r(G) + m, \quad r(H) + n, \quad r(M) + t\}, \quad (4.14)$$

$$\min_{X \in \mathbb{C}_t^{m \times n}} r[\phi(X)] = \max \{r(G) + r(H) - r(A - B - C + D), \quad r(G) + r(H) - r(M) + t, \quad r(M) - t\}. \quad (4.15)$$

Because the right-hand sides of (4.4), (4.5), (4.14) and (4.15) can be calculated exactly, these results can be taken as test examples in fixed-rank or lower-rank approximation and perturbation analysis of matrices. They can also be used to verify the correctness of various numerical algorithms in rank minimization problems occurred in recent years.

5 Concluding remarks

Closed-form formulas are established for calculating the extremal ranks in (1.7)–(1.10). This work shows a surprising fact that many matrix rank optimization problems do exist analytical solutions for calculating the extremal ranks.

Besides (1.7)–(1.10), a more popular problem is to minimize the norm of (1.6) subject to low-rank constraint

$$\text{minimize } \|A + BXC\|_F \quad \text{s.t. } X \in \mathbb{C}^{p \times q} \text{ and } r(X) \leq t, \quad (5.1)$$

see [3, 7]. So that a comparison of solutions to (1.6) and (5.1) can further be discussed.

Matrix rank optimization problem is really a fruitful research field in both matrix analysis and optimization theory. In recent years, many numerical methods were developed for matrix rank minimization problems based on approximation and iteration methods. However, there is no evidence that these numerical methods can make the matrix-rank-objective functions really attain their minimal values. Because the exact extremal ranks of $A + BXC$ can be calculated by the analytical formulas in this note, they will set a principle for verifying the correctness and effectiveness of these numerical methods.

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